MARDEN'S TAMENESS CONJECTURE: HISTORY AND APPLICATIONS

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ABSTRACT. Marden's Tameness Conjecture predicts that every hyperbolic 3-manifold with finitely generated fundamental group is homeomorphic to the interior of a compact 3-manifold. It was recently established by Agol and Calegari-Gabai. We will survey the history of work on this conjecture and discuss its many applications.

1. INTRODUCTION

In a seminal paper, published in the Annals of Mathematics in 1974, Al Marden [65] conjectured that every hyperbolic 3-manifold with finitely generated fundamental group is homeomorphic to the interior of a compact 3-manifold. This conjecture evolved into one of the central conjectures in the theory of hyperbolic 3-manifolds. For example, Marden's Tameness Conjecture implies Ahlfors' Measure Conjecture (which we will discuss later). It is a crucial piece in the recently completed classification of hyperbolic 3-manifolds with finitely generated fundamental group. It also has important applications to geometry and dynamics of hyperbolic 3-manifolds and gives important group-theoretic information about fundamental groups of hyperbolic 3-manifolds.

There is a long history of partial results in the direction of Marden's Tameness Conjecture and it was recently completely established by Agol [1] and Calegari-Gabai [27]. In this brief expository paper, we will survey the history of these results and discuss some of the most important applications.

Outline of paper: In section 2, we recall basic definitions from the theory of hyperbolic 3-manifolds. In section 3, we construct a 3-manifold with finitely generated fundamental group which is not homeomorphic to the interior of a compact 3-manifold. In section 4, we discuss some of the historical background for Marden's Conjecture and introduce the conjecture. In section 5, we introduce Thurston's notion

of geometric tameness, which turns out to be equivalent to topological tameness. In section 6, we sketch the history of partial results on the conjecture. In section 7, we give geometric applications of the Tameness Theorem, including Ahlfors' Measure Conjecture, spectral theory of hyperbolic 3-manifolds and volumes of closed hyperbolic 3-manifolds. In section 8, we discuss group-theoretic applications, including applications to the finitely generated intersection property and separability properties of subgroups of Kleinian groups. In section 9, we discuss Simon's conjecture that the interior of a cover with finitely generated fundamental group of a compact irreducible 3-manifold is topologically tame and give Long and Reid's proof of Simon's conjecture from Simon's work and the Tameness Theorem. In section 10, we explain the role of the Tameness Theorem in the classification of hyperbolic 3-manifolds with finitely generated fundamental group. In section 11, we discuss applications to the deformation theory of hyperbolic 3-manifolds.

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2. Basic Definitions

A (complete) hyperbolic 3-manifold is a complete Riemannian 3-manifold with constant sectional curvature -1. **Throughout this paper we will assume that all manifolds are orientable.** Any hyperbolic 3-manifold may be obtained as the quotient $N = \mathbb{H}^3/\Gamma$ where Γ is a group of orientation-preserving isometries acting properly discontinuously on $\mathbb{H}^3$. The group Γ is called a *Kleinian group.* (More generally, a Kleinian group is a discrete subgroup of $\mathrm{Isom}_+(\mathbb{H}^3)$. In this paper, all Kleinian groups will be assumed to be torsion-free, so that their quotient is a hyperbolic 3-manifold.)

The group $\mathrm{Isom}_+(\mathbb{H}^3)$ of orientation-preserving isometries of $\mathbb{H}^3$ is naturally identified with the group $\mathbb{PSL}_2(\mathbb{C})$ of Mobius transformations of the Riemann sphere $\widehat{\mathbb{C}}$, which we regard as the boundary at infinity of $\mathbb{H}^3$. So, if $N = \mathbb{H}^3/\Gamma$, then Γ acts also as a group of conformal automorphisms of $\widehat{\mathbb{C}}$. We divide $\widehat{\mathbb{C}}$ up into the *domain of discontinuity* $\Omega(\Gamma)$ which is the largest open subset of $\widehat{\mathbb{C}}$ on which Γ acts properly discontinuously, and its complement, $\Lambda(\Gamma)$,which is called the *limit set.*

Since Γ acts properly discontinuously on $\Omega(\Gamma)$, the quotient $\partial_c(N) = \Omega(\Gamma)/\Gamma$ is a Riemann surface, called the *conformal boundary*. One may naturally append $\partial_c(N)$ to N to obtain a 3-manifold with boundary

$$\hat{N} = N \cup \partial_c(N) = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma.$$

Of course, the nicest possible situation is that this bordification $\hat{N}$ gives a compactification of N, and N will be called *convex cocompact* if $\hat{N}$ is compact. More generally, N is said to be *geometrically finite* if $\hat{N}$ is homeomorphic to $M - P$ where M is a compact 3-manifold and P is a finite collection of disjoint annuli and tori in ∂M. (These definitions are non-classical, see Marden [65] and Bowditch [13] for a discussion of their equivalence to more standard definitions.) If Γ contains no parabolic elements, equivalently if every homotopically non-trivial simple closed curve in N is homotopic to a closed geodesic, then N is said to have *no cusps*. If N has no cusps, then it is geometrically finite if and only if it is convex cocompact. If $N = \mathbb{H}^3/\Gamma$ is geometrically finite, we will also say that the associated Kleinian group Γ is geometrically finite.

We will say that a 3-manifold is *topologically tame* if it is homeomorphic to the interior of a compact 3-manifold. It is clear from the definition we gave of geometric finiteness, that geometrically finite hyperbolic 3-manifolds are topologically tame.

If N is a hyperbolic 3-manifold, then its *convex core* $C(N)$ is the smallest convex submanifold C of N such that the inclusion of C into N is a homotopy equivalence. More concretely, $C(N)$ is the quotient $CH(\Lambda(\Gamma))/\Gamma$ of the convex hull of the limit set. Except in the special case where $\Lambda(\Gamma)$ is contained in a circle in the Riemann sphere, the convex core $C(N)$ is homeomorphic to $\hat{N}$. In particular, N is convex cocompact if and only if its convex core is compact.

Thurston [101] showed that the boundary of the convex core is a hyperbolic surface, in its intrinsic metric. There is a strong relationship between the geometry of the boundary of the convex core and the geometry of the conformal boundary. We recall that the conformal boundary admits a unique hyperbolic metric in its conformal class, called the Poincaré metric. Sullivan showed that there exists a constant K such that if N has finitely generated, freely indecomposable fundamental group, then there exists a K-bilipschitz map between $\partial C(N)$ and $\partial_c(N)$. Epstein and Marden [43] gave a careful proof of this result and showed that $K \leq 82.8$. More complicated analogues of Sullivan's result are known to hold when the fundamental group of N is not freely indecomposable, see Bridgeman-Canary [17].

3. TOPOLOGICALLY WILD MANIFOLDS

In order to appreciate Marden's Tameness Conjecture we will sketch a construction of a 3-manifold with finitely generated fundamental group which is not topologically tame.

Whitehead [106] gave an example of a simply connected (in fact, contractible) 3-manifold which is not homeomorphic to the interior of a ball. One may use Whitehead's construction to give examples of 3-manifolds with non-trivial finitely generated fundamental group which are not topologically tame. Scott and Tucker [92] give an interesting collection of examples of topologically wild 3-manifolds with a variety of properties.

The following example is essentially drawn from Tucker [104]. Whitehead's example is given as a nested union of solid tori. The fundamental group of our example will be the free group of rank two and will be obtained as a nested union of handlebodies of genus 2.

Let H denote a handlebody of genus 2 and let $g : H \to H$ be an embedding of H into the interior of H such that

(1) $g_* : \pi_1(H) \to \pi_1(H)$ is the identity map,
(2) ∂H and $g(\partial H)$ are both incompressible in $H - g(H)$ (i.e. the inclusions induce injections of $\pi_1(\partial H)$ and $\pi_1(g(\partial H))$ into $\pi_1(H - g(H))$)
(3) $\pi_1(g(\partial H))$ and $\pi_1(\partial H)$ give rise to proper subgroups of $\pi_1(H - g(H))$.

Let $B = H - g(H)$.

We then construct a manifold by letting $H_0 = H$ and $H_\infty = \bigcup H_n$ where the pair (H_n, H_{n-1}) is homeomorphic to $(H, g(H))$ for all $n \geq 1$. More concretely, we could define $H_n = H_{n-1} \cup B$ where we think of ∂H_{n-1} as the copy of ∂H in the $(n-1)^{\text{st}}$ copy of B and identify ∂H with $g(\partial H)$ by the map g. Notice that H_n is a handlebody for all n.

The inclusion of H_0 into H_∞ is a homotopy equivalence, since the inclusion of H_{n-1} into H_n is a homotopy equivalence for all n. Therefore, $\pi_1(H_\infty)$ is the free group on two generators. Note also that each H_n is a compact core for H_∞. However, one can repeatedly apply the Seifert-Van Kampen theorem to show that $H_\infty - H_0$ has infinitely generated fundamental group. It then follows that H_∞ is not topologically tame, since the complement of a compact submanifold of a topologically tame 3-manifold must have finitely generated fundamental group.

To check that $\pi_1(H_\infty - H_0)$ is infinitely generated, we observe that $\pi_1(H_n - H_0) = \pi_1(H_{n-1} - H_0) *_{\pi_1(\partial H)} \pi_1(B)$ and that $\pi_1(\partial H)$ injects into both factors. Moreover, $\pi_1(\partial H)$ is also a proper subgroup of each factor. It follows that $\pi_1(H_{n-1} - H_0)$ injects into $\pi_1(H_n - H_0)$ and that

its image is a proper subgroup. If $H_\infty - H_0$ had finitely generated fundamental group, then its generators would have representatives lying in $H_{n-1} - H_0$ for some n which would contradict the previous sentence. Alternatively, one could note that

$$\pi_1(H_\infty - H_0) = \pi_1(B) *_{\pi_1(\partial H)} \pi_1(B) *_{\pi_1(\partial H)} \pi_1(B) *_{\pi_1(\partial H)} \cdots$$

and use properties of amalgamated free products to check that this group is infinitely generated.

Tucker [104] showed that one can always detect that an irreducible 3-manifold is topologically wild by considering the fundamental group of the complement of a compact submanifold.

Theorem 3.1. (Tucker [104]) *An irreducible 3-manifold M (without boundary) is not topologically tame if and only if there exists a compact submanifold C of M such that $\pi_1(M - C)$ is infinitely generated.*

4. Prehistory and the conjecture

Hyperbolic 2-manifolds with finitely generated fundamental group are well-known to be geometrically finite. For many years, it was unclear whether the analogous statement was true for hyperbolic 3-manifolds. In 1966, Leon Greenberg [48] established the existence of hyperbolic 3-manifolds with finitely generated fundamental group which are not geometrically finite. Troels Jørgensen [57] was the first to explicitly exhibit geometrically infinite hyperbolic 3-manifolds with finitely generated fundamental group.

One early piece of evidence for Marden's Tameness Conjecture was provided by Ahlfors' Finiteness Theorem [5] which asserts that if Γ is finitely generated, then $\partial_c(N)$ is a finite collection of finite type Riemann surfaces. Alternatively, one could say that the conformal boundary has finite area in its Poincaré metric.

On the topological side, Peter Scott [89] showed that any 3-manifold M with finitely generated fundamental group contains a *compact core*, i.e. a compact submanifold C such that the inclusion of C into M induces an isomorphism from $\pi_1(C)$ to $\pi_1(M)$. If M is a hyperbolic 3-manifold, or more generally if M is irreducible, one may assume that the inclusion of C into M is a homotopy equivalence. This implies, in particular, that finitely generated 3-manifold groups are actually finitely presented. It follows from work of McCullough, Miller and Swarup [71] that if M is irreducible and topologically tame, then it is homeomorphic to the interior of its compact core.

In 1974, Al Marden published a long paper [65] which was the first papers to bring to bear the classical results of 3-manifold topology on

the study of hyperbolic 3-manifolds. Previously, most of the work on hyperbolic 3-manifolds, was done by considering the actions of their fundamental groups on the Riemann sphere. In the appendix of this paper, Marden asked two prescient questions which we will rephrase in our language.

Marden's first question: *If N is a hyperbolic 3-manifold with finitely generated fundamental group, is N topologically tame?*

Marden's second question: *Is there a necessary and sufficient condition which guarantees that a compact 3-manifold M is hyperbolizable? (A compact manifold is hyperbolizable if there admits a complete hyperbolic metric on the interior of M.)*

The first question became known as Marden's Tameness Conjecture, while the second question foreshadows Thurston's Geometrization Conjecture. In particular, Thurston's Geometrization Conjecture predicted that a compact 3-manifold is hyperbolizable if it is irreducible, atoroidal and its fundamental group is not virtually abelian. We recall that a compact 3-manifold M is *irreducible* if every embedded 2-sphere bounds a ball and is called *atoroidal* if $\pi_1(M)$ does not contain a free abelian subgroup of rank two. Thurston established his hyperbolization conjecture whenever M is Haken, e.g. whenever M has non-empty boundary. See Morgan [76] or Kapovich [58] for extensive discussions of Thurston's Hyperbolization Theorem. Perelman [84, 85] recently gave a proof of Thurston's entire Geometrization Conjecture. (See Kleiner-Lott [63], Morgan-Tian [77] and Cao-Zhu [38] for expositions of Perelman's work.)

5. Geometric tameness

In Thurston's work on the Geometrization Conjecture he developed a notion called geometric tameness. His definition only worked in the setting of hyperbolic 3-manifolds with freely indecomposable fundamental group. We will give a definition of geometric tameness, first introduced in [29], which works in a more general setting.

In order to motivate the definition we will consider a specific example of a geometrically infinite manifold. Thurston [102] showed that any atoroidal 3-manifold which fibers over the circle, whose fundamental group is not virtually abelian, is hyperbolizable. (Thurston was inspired by Jørgensen's example [57] which was the cover of a hyperbolic 3-manifold which fibers over the circle.) Let M be a closed hyperbolic 3-manifold which fibers over the circle with fiber the closed surface S and let $\hat{M}$ be the regular cover associated to $\pi_1(S)$. The fibered manifold M is obtained from $S \times [0,1]$ by gluing $S \times \{0\}$ to $S \times \{1\}$ by a

homeomorphism $\varphi : S \to S$ and the manifold $\hat{M}$ is obtained from infinitely many copies of $S \times [0, 1]$ stacked one on top of the other. So, $\hat{M}$ is homeomorphic to $S \times \mathbb{R}$ and the group of covering transformations is generated by $\hat{\varphi} : S \times \mathbb{R} \to S \times \mathbb{R}$ where $\hat{\varphi}(x, t) = (\varphi(x), t+1)$. One sees that the ends of $\hat{M}$ grow linearly, which is surprising for a hyperbolic manifold. In particular, if one considers a minimal surface in M in the homotopy class of S, its pre-image in $\hat{M}$ is an infinite family of surfaces exiting both ends of $\hat{M}$. The key property of these surfaces is that they have curvature ≤ -1. Thurston realized that the existence of such a family of surfaces was both widespread and quite useful.

In our definition of geometric tameness, we will make use of simplicial hyperbolic surfaces. We will give a careful definition of a simplicial hyperbolic surface $f : S \to N$, but we first note that the key issue is that the induced metric on S has curvature ≤ -1 (in the sense of Alexandrov.) Thurston [101] originally made use of pleated surfaces, Minsky [72] showed that one can use harmonic maps, Bonahon [12] pioneered the use of simplicial hyperbolic surfaces, Calegari and Gabai [27] used shrinkwrapped surfaces and Soma [94] used ruled wrappings (which are a simplicial analogue of shrinkwrapped surfaces.)

A map $f : S \to N$ is a *simplicial hyperbolic surface* if there exists a triangulation T of S such that f maps faces of T to totally geodesic immersed triangles in N and the total angle of the triangles about any vertex adds up to at least 2π. We note that we allow our triangulations to have the property that vertices and edges of faces may be identified. For example, one can obtain a triangulation of a torus with two faces by adding the diagonal to the usual square gluing diagram. Simplicial hyperbolic surfaces share many useful properties with actual hyperbolic surfaces. For example, their area is bounded above by $2\pi|\chi(S)|$ and the diameter of each component of their "thick part" is uniformly bounded.

Let N be a hyperbolic 3-manifold with finitely generated fundamental group. For the purposes of simplifying the definition we will assume that N has no cusps. Let C be a compact core for N. The ends of N may be identified with the components of $N - C$. For our purposes, a neighborhood U of an end E is a subset of E such that $E - U$ has compact closure. We will say that an end is *geometrically finite* if it has a neighborhood disjoint from the convex core. We say that an end is *simply degenerate* if it has a neighborhood U which is homeomorphic to $S \times (0, \infty)$ for some closed surface S and there exists a sequence $\{f_n : S \to U\}$ of simplicial hyperbolic surfaces, such that

(1) given any compact subset K of N, $f_n(S) \cap K$ is empty for all but finitely many n, and

(2) for all n, f_n is homotopic, within U, to the map $h_1 : S \to U$ given by $h_1(x) = (x, 1)$.

A hyperbolic 3-manifold with finitely generated fundamental group is said to be *geometrically tame* if each of its ends is either geometrically finite or simply degenerate.

It is not difficult to show that any geometrically finite end has a neighborhood homeomorphic to $S \times (0, \infty)$ for some closed surface S. Therefore, one easily observes that geometrically tame hyperbolic 3-manifolds are topologically tame. Thurston originally gave a much weaker definition of geometric tameness in the setting of hyperbolic 3-manifolds with freely indecomposable fundamental group. In this setting, Thurston [101] was able to show that his weaker definition of geometric tameness implied topological tameness and in fact implied the definition given here.

6. History

In this section, we will give a brief history of the partial results on Marden's Tameness Conjecture leading up to its final solution. Many of these results involved looking at limits of geometrically finite or topologically tame hyperbolic 3-manifolds. Readers who are not interested in the historical development may prefer to skip ahead to the next section.

There are two types of convergence, algebraic and geometric, which play a prominent role in the theory of hyperbolic 3-manifolds. A sequence $\{\rho_n : G \to \mathbb{PSL}_2(\mathbb{C})\}$ of discrete faithful representations is said to *converge algebraically* to $\rho : G \to \mathbb{PSL}_2(\mathbb{C})$ if it converges in the compact-open topology on $\mathrm{Hom}(G, \mathbb{PSL}_2(\mathbb{C}))$. If G is not virtually abelian (i.e. does not contain a finite index abelian subgroup), then ρ is also discrete and faithful (see Chuckrow [40] and Jørgensen [56].) A sequence of Kleinian groups $\{\Gamma_n\}$ is said to *converge geometrically* to a Kleinian group Γ if every $\gamma \in \Gamma$ arises as a limit of a sequence $\{\gamma_n \in \Gamma_n\}$ and every limit β of a sequence of elements $\{\gamma_{n_k}\}$ in a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ lies in Γ. We say that a sequence of hyperbolic 3-manifolds $\{N_n\}$ converges geometrically to N if one may write $N_n = \mathbb{H}^3/\Gamma_n$ and $N = \mathbb{H}^3/\Gamma$ so that $\{\Gamma_n\}$ converges geometrically to Γ. This implies, see [33] or [9], that $\{N_n\}$ converges in the sense of Gromov to N, i.e. "larger and larger" subsets of N_n look "increasingly like" large subsets of N (as n goes to ∞.) A sequence $\{\rho_n : G \to \mathbb{PSL}_2(\mathbb{C})\}$ of discrete faithful representations is said to *converge strongly* to $\rho : G \to \mathbb{PSL}_2(\mathbb{C})$ it it converges algebraically and $\{\rho_n(G)\}$ converges geometrically to $\rho(G)$.

Thurston [101] proved the following theorem concerning limits of geometrically finite hyperbolic 3-manifolds with freely indecomposable fundamental group. He used his theorem in the proof of his geometrization theorem.

Theorem 6.1. (Thurston [101]) *Let G be a finitely generated, torsion-free, freely indecomposable, non-abelian group. If $\{\rho_n : G \to \mathbb{PSL}_2(\mathbb{C})\}$ is a sequence of discrete, faithful, geometrically tame representations converging algebraically to $\rho : G \to \mathbb{PSL}_2(\mathbb{C})$ such that $\rho_n(g)$ is parabolic for some n if and only if $\rho(g)$ is parabolic, then $\{\rho_n\}$ converges strongly to ρ and $\rho(G)$ is geometrically tame.*

In a breakthrough paper, Bonahon [12] proved that all hyperbolic 3-manifolds with freely indecomposable, finitely generated fundamental group are geometrically tame, and hence topologically tame.

Theorem 6.2. (Bonahon[12]) *If N is a hyperbolic 3-manifold with finitely generated, freely indecomposable group then N is geometrically tame.*

Canary [29] used Bonahon's work to show that topological tameness and geometric tameness are equivalent notions.

Theorem 6.3. (Canary [29]) *Let N be a hyperbolic 3-manifold with finitely generated fundamental group. Then, N is topologically tame if and only if N is geometrically tame.*

Canary-Minsky [36] and Ohshika [81] showed that, in the absence of cusps, strong limits of topologically tame hyperbolic 3-manifolds are themselves topologically tame. In order to more easily state this result, we define a representation $\rho : G \to \mathbb{PSL}_2(\mathbb{C})$ to be *purely hyperbolic* if $\rho(G)$ contains no parabolic elements.

Theorem 6.4. (Canary-Minsky [36],Ohshika[81]) *Let G be a finitely generated, torsion-free, non-abelian group. If $\{\rho_n : G \to \mathbb{PSL}_2(\mathbb{C})\}$ is a sequence of discrete, faithful, topologically tame, purely hyperbolic representations converging strongly to a purely hyperbolic representation $\rho : G \to \mathbb{PSL}_2(\mathbb{C})$, then $\rho(G)$ is topologically tame.*

By applying results of Anderson-Canary [6] or Ohshika [81] one can often guarantee that when both the approximates and the limit in an algebraically convergent sequence are purely hyperbolic, then the sequence converges strongly.

Corollary 6.5. *Let G be a finitely generated, torsion-free, non-abelian group. If $\{\rho_n : G \to \mathbb{PSL}_2(\mathbb{C})\}$ is a sequence of discrete, faithful,*

topologically tame, purely hyperbolic representations converging alge-braically to a purely hyperbolic representation $\rho : G \to \mathbb{PSL}_2(\mathbb{C})$ *such that either* $\Lambda(\rho(G)) = \widehat{\mathbb{C}}$ *or* G *is not a free product of surface groups, then* $\rho(G)$ *is topologically tame.*

Evans [44] was able to significantly weaken the assumption that the representations are purely hyperbolic.

Theorem 6.6. (Evans[44]) *Let* G *be a finitely generated, torsion-free, non-abelian group. If* $\{\rho_n : G \to \mathbb{PSL}_2(\mathbb{C})\}$ *is a sequence of discrete, faithful, topologically tame representations converging algebraically to* $\rho : G \to \mathbb{PSL}_2(\mathbb{C})$ *such that*

(1) *either* $\Lambda(\rho(G)) = \widehat{\mathbb{C}}$ *or* G *is not a free product of surface groups, and*

(2) *if* $g \in G$ *and* $\rho(g)$ *is parabolic, then* $\rho_n(g)$ *is parabolic for all* n,

then $\rho(G)$ *is topologically tame.*

Juan Souto [96] showed that if N can be exhausted by compact cores then it is topologically tame. (In fact, he proves somewhat more, but we will just state the simpler version.) Kleineidam and Souto [62] used Souto's result to show that if a Masur domain lamination on a boundary component is not realizable, then the corresponding end is tame (see [62] for definitions). Souto's work was quite influential in the later solutions of Marden's Tameness Conjecture.

Theorem 6.7. (Souto[96]) *If* N *is a hyperbolic 3-manifold with finitely generated fundamental group and* $N = \bigcup C_i$ *where* C_i *is a compact core for* N *and* $C_i \subset C_{i+1}$ *for all* i, *then* N *is topologically tame*

Brock, Bromberg, Evans and Souto [19] were able to show that "most" algebraic limits of geometrically finite hyperbolic 3-manifolds are topologically tame.

Theorem 6.8. (Brock-Bromberg-Evans-Souto [19]) *Let* G *be a finitely generated, torsion-free, non-abelian group. If* $\{\rho_n : G \to \mathbb{PSL}_2(\mathbb{C})\}$ *is a sequence of discrete, faithful, geometrically finite representations converging algebraically to* $\rho : G \to \mathbb{PSL}_2(\mathbb{C})$ *such that either*

(1) $\Lambda(\rho(G)) = \widehat{\mathbb{C}}$,

(2) G *is not a free product of surface groups, or*

(3) $\{\rho_n(G)\}$ *converges geometrically to* $\rho(G)$,

then $\rho(G)$ *is topologically tame.*

Brock and Souto [23] were able to resolve the remaining cases to prove that all algebraic limits of geometrically finite hyperbolic 3-manifolds are topologically tame.

Theorem 6.9. (Brock-Souto[23]) *Let G be a finitely generated, torsion-free, non-abelian group. If $\{\rho_n : G \to \mathbb{PSL}_2(\mathbb{C})\}$ is a sequence of discrete, faithful, geometrically finite representations converging algebraically to $\rho : G \to \mathbb{PSL}_2(\mathbb{C})$, then $\rho(G)$ is topologically tame.*

In 2004, Ian Agol and the team of Danny Calegari and David Gabai announced proofs of Marden's Tameness Conjecture.

Tameness Theorem: (Agol[1],Calegari-Gabai[27]) *If N is a hyperbolic 3-manifold with finitely generated fundamental group, then N is topologically tame.*

Soma [94] later simplified the proof by combining ideas of Agol and Calegari-Gabai. Bowditch [14] also gives an account of the result using ideas of Agol, Calegari-Gabai and Soma. He also describes how to generalize the proof for manifolds of pinched negative curvature and uses it to prove an analogue of Ahlfors' Finiteness Theorem in this setting. Choi [39] has an alternate approach to the proof.

7. Geometric Applications

In the remaining sections we will collect some of the major applications and consequences of the solution of Marden's Tameness Conjecture.

Geometric tameness gives one strong control on the geometry of ends. One manifestation of this control is a minimum principle for superharmonic functions which Thurston [101] established for geometrically tame hyperbolic 3-manifolds with incompressible boundary and Canary [29] generalized to the setting of topologically tame hyperbolic 3-manifolds. Given the Tameness Theorem one need only require that our manifold have finitely generated fundamental group.

Theorem 7.1. *(Thurston [101],Canary[29]) Let N be a hyperbolic 3-manifold with finitely generated fundamental group. If $h : N \to (0, \infty)$ is a positive superharmonic function, i.e.* $\operatorname{div}(\operatorname{grad} h) \geq 0$, *then*

$$\inf_{C(N)} h = \inf_{\partial C(N)} h.$$

In particular, if $C(N) = N$, then h is constant.

The main idea of the proof here is to consider the flow generated by $-\operatorname{grad} h$, i.e. the flow in the direction of maximal decrease. The fact that h is superharmonic guarantees that this flow is volume non-decreasing. The fact that h is positive guarantees that the flow moves more and more slowly as one progress. Neighborhoods of radius one of our simplicial hyperbolic surfaces have bounded volume, so act as narrows for the flow. It follows that almost every flow line starting

in $C(N)$ must exit the convex core through its boundary, rather than flowing out one of the simply degenerate ends.

One of the most important applications of this minimum principle is a solution of Ahlfors' Measure Conjecture. Ahlfors [4] proved that a limit set of a geometrically finite hyperbolic 3-manifold either has measure zero or is the entire Riemann sphere. He conjectured that this would hold for all hyperbolic 3-manifolds with finitely generated fundamental group.

Corollary 7.2. *If $N = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold with finitely generated fundamental group, then either $\Lambda(\Gamma)$ has measure zero in $\widehat{\mathbb{C}}$ or $\Lambda(\Gamma) = \widehat{\mathbb{C}}$. Moreover, if $\Lambda(\Gamma) = \widehat{\mathbb{C}}$ then Γ acts ergodically on $\widehat{\mathbb{C}}$, i.e. if $A \subset \widehat{\mathbb{C}}$ is measurable and Γ-invariant, then A has either measure zero or full measure.*

The proof of this corollary follows the same outline as Ahlfors' original proof. We suppose that $\Lambda(\Gamma) \neq \widehat{\mathbb{C}}$ and has positive measure and consider the harmonic function $\tilde{h} : \mathbb{H}^3 \to (0,1)$ given by letting $\tilde{h}(x)$ be the proportion of geodesic rays emanating from x which end at points in the limit set $\Lambda(\Gamma)$. The function $\tilde{h}$ is Γ-invariant, so descends to a harmonic function $h : N \to (0,1)$. It is clear that $h \leq \frac{1}{2}$ on $N - C(N)$, so the minimum principle applied to $1 - h$ implies that $h \leq \frac{1}{2}$ on all of N. Therefore, $\tilde{h} \leq \frac{1}{2}$ on $\mathbb{H}^3$. On the other hand, as x approaches a point of density of $\Lambda(\Gamma)$ along a geodesic, it is clear that $h(x)$ must approach 1, so we have achieved a contradiction. (To establish the ergodicity of the action in the case that $\Lambda(\Gamma) = \widehat{\mathbb{C}}$, one assumes that A is a Γ-invariant set which has neither full or zero measure and studies the function $\tilde{h}$ which is the proportion of rays emanating from a point which end in A.)

Another immediate consequence of this minimum principle is a characterization of which hyperbolic 3-manifolds admit non-constant positive superharmonic functions.

Corollary 7.3. *Let $N = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold with finitely generated fundamental group. The manifold N is strongly parabolic (i.e. admits no non-constant positive superharmonic functions) if and only if $\Lambda(\Gamma) = \widehat{\mathbb{C}}$.*

Sullivan [97] showed that the geodesic flow of N is ergodic if and only if it admits a (positive) Green's function, so one can also completely characterize when the geodesic flow of N is ergodic.

Corollary 7.4. *Let $N = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold with finitely generated fundamental group. The geodesic flow of N is ergodic if and only if $\Lambda(\Gamma) = \widehat{\mathbb{C}}$.*

Another collection of geometric applications of topological tameness involve the Hausdorff dimension of the limit set and the bottom of the spectrum of the Laplacian. Patterson [83] and Sullivan [99] showed that there are deep relationships between these two quantities. In particular, they showed that if $N = \mathbb{H}^3/\Gamma$ is geometrically finite, then

$$\lambda_0(N) = D(\Lambda(\Gamma))(2 - D(\Lambda(\Gamma))$$

unless $D(\Lambda(\Gamma) < 1$ in which case $\lambda_0(N) = 1$. Here, $D(\Lambda(\Gamma))$ denotes the Hausdorff dimension of the limit set and $\lambda_0(N) = \inf \mathrm{spec}(-\mathrm{div}(\mathrm{grad}))$ is the bottom of the spectrum of the Laplacian.

Sullivan [99] and Tukia [105] showed that if N is geometrically finite and has infinite volume, then $\lambda_0(N) > 0$. Canary [28] proved that if N is topologically tame and geometrically infinite, then $\lambda_0(N) = 0$. (One does this by simply using the simplicial hyperbolic surfaces exiting the end to show that the Cheeger constant of a geometrically infinite manifold is 0.)

Theorem 7.5. (Sullivan [99],Tukia[105],Canary[28]) *Let $N = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold with finitely generated fundamental group. Then $\lambda_0(N) = 0$ if and only if either N has finite volume or is geometrically infinite.*

Bishop and Jones [11] showed that geometrically infinite hyperbolic 3-manifolds have limit sets of Hausdorff dimension 2 without making use of tameness. Combining all the results we have mentioned one gets the following result.

Corollary 7.6. *Let $N = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold with finitely generated fundamental group. Then,*

$$\lambda_0(N) = D(\Lambda(\Gamma))(2 - D(\Lambda(\Gamma))$$

unless $D(\Lambda(\Gamma) < 1$ in which case $\lambda_0(N) = 1$.

Remark: The Hausdorff dimension of the limit set can only be less than 1 if Γ is a geometrically finite free group and it can only be equal to 1 if it is a geometrically finite free group or a surface group which is conjugate to a subgroup of $\mathbb{PSL}_2(\mathbb{R})$ (see Braam [16], Canary-Taylor [37] and Sullivan [97]).

Marc Culler, Peter Shalen and their co-authors have engaged in an extensive study of the relationship between the topology and the volume of a hyperbolic 3-manifold. At the core of this study is a quantitative generalization of the Margulis lemma which they originally established for purely hyperbolic, geometrically finite, free groups of rank two (and their limits) in [41] and generalized to free groups of all ranks in [7]. The Tameness Theorem (and the Density Theorem which we will discuss later) allows us to remove the tameness and the hyperbolicity assumptions.

Theorem 7.7. (Anderson-Canary-Culler-Shalen [7]) *Let* Γ *be a Kleinian group freely generated by elements* $\{\gamma_1, \ldots, \gamma_k\}$. *If* $z \in \mathbb{H}^3$, *then*

$$\sum_{i=1}^{k} \frac{1}{1 + e^{d(z,\gamma_i(z))}} \leq \frac{1}{2}.$$

In particular there is some $i \in \{1, \ldots, k\}$ *such that*

$$d(z, \gamma_i(z)) \geq \log(2k - 1).$$

Culler and Shalen have an extensive body of work making use of the above estimate to obtain volume estimates.

In some cases, the Tameness Theorem yields immediate improvements of the results in this program. For example, in the following result one originally also had to assume that every 3-generator subgroup of $\pi_1(M)$ is topologically tame.

Theorem 7.8. (Anderson-Canary-Culler-Shalen [7]) *If* N *is a closed hyperbolic 3-manifold such that every 3-generator subgroup of* $\pi_1(N)$ *is free, then the volume of* N *is at least 3.08.*

As another example of the type of results that Culler and Shalen obtain, we state one of their recent theorems, whose proof makes use of the above estimate and the Tameness Theorem.

Theorem 7.9. (Culler-Shalen [42]) *If* N *is a closed hyperbolic 3-manifold and* $H_1(M, \mathbf{Z}_2) \geq 8$, *then the volume of* N *is at least 3.08.*

8. GROUP-THEORETIC APPLICATIONS

The resolution of Marden's Tameness Conjecture allows one to improve many previous results concerning group-theoretic properties of hyperbolic 3-manifolds. The main tool here is a corollary of the Covering Theorem which allows one to completely characterize geometrically infinite covers of a finite volume hyperbolic 3-manifold. The applications will be to the finitely generated intersection property, separability

properties of subgroups of Kleinian groups, the pro-normal topology on a Kleinian group and to commensurators of subgroups of Kleinian groups.

The Covering Theorem produces restrictions on how a hyperbolic 3-manifold with a simply degenerate end can cover another hyperbolic 3-manifold. It was proved in the case that the covering manifold is geometrically tame with incompressible boundary by Thurston [101] and in the case where the covering manifold is allowed to be topologically tame by Canary [31]. (For versions where the base space is allowed to be an orbifold, see Agol [1] and Canary-Leininger [34].)

Covering Theorem: (Thurston[101], Canary [31]) *Let $\hat{N}$ be a hyperbolic 3-manifold with finitely generated fundamental group which covers another hyperbolic 3-manifold N by a local isometry $p : \hat{N} \to N$. If $\hat{E}$ is a geometrically infinite end of $\hat{N}$ then either*

a) $\hat{E}$ has a neighborhood $\hat{U}$ such that p is finite-to-one on $\hat{U}$, or

b) N has finite volume and has a finite cover N' which fibers over the circle such that if N_S denotes the cover of N' associated to the fiber subgroup then $\hat{N}$ is finitely covered by N_S. Moreover, if $\hat{N} \neq N_S$, then $\hat{N}$ is homeomorphic to the interior of a twisted I-bundle which is doubly covered by N_S.

Remark: The statement above assumes that $\hat{N}$ has no cusps. If $\hat{N}$ is allowed to have cusps, then one must consider ends of $\hat{N}^0$, which is N with the cuspidal portions of its thin part removed. One can define simply degenerate and geometrically infinite ends in this context and the statement is essentially the same.

In the case that $N = \mathbb{H}^3/\Gamma$ is a finite volume hyperbolic 3-manifold, we see that all geometrically infinite, finitely generated subgroups of Γ are associated to fibre subgroups of finite covers of N which fiber over the circle. A subgroup $\hat{\Gamma}$ of Γ is said to be a *virtual fiber subgroup* if there exist finite index subgroups Γ_0 of Γ and $\hat{\Gamma}_0$ of $\hat{\Gamma}$ such that $N_0 = \mathbb{H}^3/\Gamma_0$ fibers over the circle and $\hat{\Gamma}_0$ corresponds to the fiber subgroup. Corollary 8.1 is the key tool in many of the group-theoretic applications of Marden's Tameness Conjecture.

Corollary 8.1. *If $N = \mathbb{H}^3/\Gamma$ is a finite volume hyperbolic 3-manifold and $\hat{\Gamma}$ is a finitely generated subgroup of Γ, then $\hat{\Gamma}$ is either geometrically finite or a virtual fiber subgroup.*

Thurston (see [30] for a proof) had earlier proved, using Ahlfors' Finiteness Theorem [5], that a cover of an infinite-volume geometrically

finite hyperbolic 3-manifold is geometrically finite if it has finitely generated fundamental group. More generally, one may use the covering theorem to completely describe exactly which covers of a hyperbolic 3-manifold with finitely generated fundamental group are geometrically finite (see [31]).

Corollary 8.1 is related to a question of Thurston.

Question 8.2. (Thurston) *Does every finite volume hyperbolic 3-manifold admit a finite cover which fibers over the circle?*

Ian Agol [2] has recently established that large classes of finite volume hyperbolic 3-manifolds have finite covers which fiber over the circle. In particular, manifold covers of reflection orbifolds and arithmetic hyperbolic orbifolds defined by a quadratic form are covered by his methods.

We first focus on the finitely generated intersection property. A group G is said to have the *finitely generated intersection property* if whenever H and H' are finitely generated subgroups of G, then $H \cap H'$ is finitely generated. Susskind [100] proved that the intersection of two geometrically finite subgroups of a Kleinian group is geometrically finite. In combination with Thurston's proof that any finitely generated subgroup of a co-infinite volume geometrically finite Kleinian group is geometrically finite, this establishes that co-infinite volume geometrically finite Kleinian groups have the finitely generated intersection property. If we combine this with Thurston's Hyperbolization Theorem for Haken 3-manifolds we obtain the following theorem of Hempel:

Theorem 8.3. (Hempel [50]) *Let M be a compact, atoroidal, irreducible 3-manifold with a non-toroidal boundary component. Then $\pi_1(M)$ has the finitely generated intersection property.*

It is well-known, see Jaco [53] for example, that hyperbolic 3-manifolds which fiber over the circle do not have the finitely generated intersection property. However, combining Susskind's result with Corollary 8.1 we see that finite volume hyperbolic 3-manifolds have the finitely generated intersection property if and only if they do not have a finite cover which fibers over the circle. Again, combining with the resolution of Thurston's Geometrization Conjecture, we get the following purely topological statement.

Theorem 8.4. *Let M be a compact, atoroidal, irreducible 3-manifold whose fundamental group is not virtually abelian. Then $\pi_1(M)$ has the finitely generated intersection property if and only if M does not have a finite cover which fibers over the circle.*

Remark: See Soma [95] for a discussion of the finitely generated intersection property for geometric manifolds which are not hyperbolic.

There are also a number of applications of Marden's Tameness Conjecture to separability properties of fundamental groups of hyperbolic 3-manifolds. If G is a group, and H a subgroup of G, then H is said to be *separable* in G if for every $g \in G \setminus H$, there is a subgroup K of finite index in G such that $H \subset K$ but $g \notin K$. G is said to be *LERF* if every finitely generated subgroup is separable. This condition is a strengthening of residual finiteness as a group is residually finite if and only if the trivial subgroup is separable. The main motivation for studying this property comes from low-dimensional topology. If the fundamental group of an irreducible 3-manifold contains a separable surface subgroup, then one can find a finite cover which contains an embedded incompressible surface and hence is Haken.

Scott [90, 91] showed that all Seifert fibered manifolds have LERF fundamental groups. However, Rubinstein and Wang [88] showed that there are graph manifolds whose fundamental groups are not LERF (and in fact contain surface subgroups which are not separable.)

A Kleinian group G is said to be GFERF if all geometrically finite subgroups are separable. Since virtually fibered subgroups are easily seen to be separable, Corollary 8.1 implies that all GFERF Kleinian groups are in fact LERF. Many examples of Kleinian groups are known to be GRERF, and hence LERF. Gitik [45] proved that fundamental groups of a large class of infinite volume hyperbolic 3-manifolds are LERF and also produced large classes of closed hyperbolic 3-manifolds which are GFERF. Wise [107] exhibited a class of non-postively curved 2-complexes whose fundamental groups are separable for all quasiconvex subgroups. As one example, he shows that the fundamental group of the figure eight knot complement (which is hyperbolic) is GFERF. Agol, Long and Reid [3] showed that the Bianchi groups $\mathrm{PSL}(2, \mathcal{O}_d)$ are GFERF. As a consequence they show that there are infinitely many hyperbolic links in S^3 whose fundamental group are GFERF, e.g. the figure eight knot and the Borromean rings. They also give examples of closed hyperbolic 3-manifolds whose fundamental groups are GFERF, e.g. the Seifert-Weber dodecahedral space.

Glasner, Souto and Storm [47] applied the solution of Marden's Tameness Conjecture and the Covering Theorem to the pro-normal topology on fundamental groups of finite volume hyperbolic 3-manifolds. We refer the reader to [47] for the definition of the pro-normal topology on the group. However, we recall that a subgroup is open if and only if it contains a non-trivial normal subgroup and it is closed if and only

if it is an intersection of open subgroups. The pro-normal topology is not always defined, so they must also establish that the pro-normal topology is well-defined.

Theorem 8.5. (Glasner-Souto-Storm[47]) *Let N be a finite volume hyperbolic 3-manifold. The pro-normal topology on $\pi_1(N)$ is well-defined and every finitely generated subgroup H of $\pi_1(N)$ is closed in this topology. Moreover, if H has infinite index in $\pi_1(N)$, then it is the intersection of open subgroups strictly containing H.*

We say that a subgroup H of a group G is *maximal* if it is not strictly contained in a proper subgroup of G. It is clear, for example, that any subgroup of index 2 is maximal. Margulis and Soifer [66] proved that every finitely generated linear group which is not virtually solvable contains a maximal subgroup of infinite index. It is natural to ask whether such a such a subgroup can be can be finitely generated.

A subgroup is called *pro-dense* if it is dense in the pro-normal topology and Gelander and Glasner [46] have established a result guaranteeing that fundamental groups of hyperbolic 3-manifolds admit pro-dense subgroups.

As a corollary of Theorem 8.5, Glasner, Souto and Storm showed that all maximal subgroups of infinite index and all pro-dense subgroups are infinitely generated.

Corollary 8.6. (Glasner-Souto-Storm[47]) *Let N be a finite volume hyperbolic 3-manifold. If H is a maximal subgroup of infinite index or a pro-dense subgroup, then H is infinitely generated.*

Remark: Their results also apply to fundamental groups of finite volume hyperbolic orbifolds.

Alan Reid showed that one can use the Tameness Theorem and Corollary 8.1 to characterize when the commensurator of a subgroup of the fundamental group of a finite volume hyperbolic 3-manifold has finite index in the entire group. We include the proof of Reid's result with his kind permission.

We recall that if H is a subgroup of a group G, then commensurator of H in G is defined to be

$$Comm_G(H) = \{g \in G \mid gHg^{-1}\text{is commensurable with } H\}.$$

(We recall that H and J are *commensurable* subgroups of G if $H \cap J$ has finite index in both H and J.) Notice that if G_0 has finite index in G, H_0 has finite index in H, and $H_0 \subset G_0$, then $Comm_{G_0}H_0 \subset Comm_G H$.

Theorem 8.7. (Reid) *Let M be a finite volume hyperbolic 3-manifold and let H be an infinite index, finitely generated subgroup of $\pi_1(M)$.*

Then $Comm_{\pi_1(M)}H$ has finite index in $\pi_1(M)$ if and only if H is a virtual fiber subgroup.

Proof. First suppose that $M = \mathbb{H}^3/\Gamma$ and H is a virtual fiber subgroup of Γ. Then there exist finite index subgroups $H_0 \subset H$ and $\Gamma_0 \subset \Gamma$ such that the cover M_0 of M associated to Γ_0 is a finite volume hyperbolic 3-manifold which fibers over the circle and H_0 is the subgroup associated to the fiber. Since H_0 is normal in Γ_0,

$$\Gamma_0 = Comm_{\Gamma_0}H_0 \subset Comm_\Gamma H.$$

If H is not a virtual fiber subgroup, then Corollary 8.1 implies that H is geometrically finite. If $\gamma \in Comm_\Gamma H$, then $H \cap \gamma H \gamma^{-1}$ has finite index in both H and $\gamma H \gamma^{-1}$. It follows that

$$\Lambda(H \cap \gamma H \gamma^{-1}) = \Lambda(H) = \Lambda(\gamma H \gamma^{-1}).$$

Since $\Lambda(\gamma H \gamma^{-1}) = \gamma(\Lambda(H))$, we see that $\Lambda(H)$ is invariant under $Comm_\Gamma H$, so $\Lambda(Comm_\Gamma H) \subset \Lambda(H)$. However, since H has infinite index in Γ, $\mathbb{H}^3/H$ has infinite volume and $\Lambda(H) \neq \widehat{\mathbb{C}}$. Since $\Lambda(\Gamma) = \widehat{\mathbb{C}}$, $Comm_\Gamma H$ must have infinite index in Γ. $\qquad\square$

Remark: In fact, one can apply Ahlfors' Finiteness Theorem here to see that if H is not a virtual fiber subgroup, then H has finite index in $Comm_\Gamma H$.

9. Simon's Conjecture

Marden's Tameness Conjecture is clearly closely related to Simon's Conjecture on covers of compact 3-manifolds, which we state in a slightly simpler form.

Conjecture 9.1. (Simon [93]) *Let M be a compact irreducible 3-manifold and let N be a cover of M with finitely generated fundamental group, then* int(N) *is topologically tame.*

Simon [93] originally proved his conjecture for covers associated to peripheral subgroups of the 3-manifold group (i.e. subgroups of the image of the fundamental group of a boundary component in the 3-manifold.) If one combines Thurston's proof that all covers (with finitely generated fundamental group) of a geometrically finite infinite volume hyperbolic 3-manifold are geometrically finite with his Hyperbolization Theorem for Haken manifolds, then one verifies Simon's conjecture for all compact, irreducible, atoroidal 3-manifolds with a non-toroidal boundary component.

The solution of Marden's Tameness Conjecture obviously implies Simon's conjecture for all compact hyperbolizable 3-manifolds. Long

and Reid pointed out that one can combine this with Simon's work to establish Simon's conjecture for all manifolds which admit a geometric decomposition. Since their argument has not appeared we will give it here.

Long and Reid's proof that Marden's Conjecture implies Simon's conjecture for manifolds which admit a geometric decomposition: We first suppose that M itself is a geometric manifold. If M is hyperbolic, then Simon's Conjecture follows directly from the Tameness Theorem. If M has SOL geometry, then M has the finitely generated intersection property (see Soma [95]), so Simon's Conjecture follows from Theorem 3.7 in Simon [93]. In the remaining cases, M is Seifert-fibered. We may clearly assume that $\pi_1(M)$ is infinite. If M admits a $S^2 \times \mathbb{R}$-structure, then it is finitely covered by $S^2 \times S^1$ and Simon's Conjecture is easily verified. In all other cases, M contains an immersed incompressible torus.

Since $\pi_1(M)$ is LERF, we can find a finite cover $\hat{M}$ of M which containes an embedded non-separating torus (see Scott [90, 91].) Thus, $\hat{M}$ is a union of $T^2 \times I$ and M_0, where M_0 is a compact Seifert fibered space with non-empty boundary. Simon's conjecture holds for M_0, by Corollary 3.3 of [93], since it has a finite cover of the form $F \times S^1$ where F is a compact surface, and it clearly holds for $T^2 \times I$. Corollary 3.2 of [93] then shows that Simon's conjecture holds for $\hat{M}$ and hence for M.

Since we have established Simon's conjecture for all the pieces in a manifold which admits a geometric decomposition, we can again apply Corollary 3.2 of Simon [93] to complete the proof for compact irreducible manifolds which admit a geometric decomposition.

The recent resolution of Thurston's Geometrization Conjecture then allows us to conclude that Simon's Conjecture holds for all compact irreducible 3-manifolds.

Theorem 9.2. *Let M be a compact irreducible 3-manifold and let N be a cover of M with finitely generated fundamental group, then $\mathrm{int}(N)$ is topologically tame.*

10. Classification of hyperbolic 3-manifolds

In 2003, Brock, Canary and Minsky [75, 21, 22] announced the proof of Thurston's Ending Lamination Conjecture for topologically tame hyperbolic 3-manifolds. This conjecture gives a complete classification up to isometry of topologically tame hyperbolic 3-manifolds in terms

of their homeomorphism types and their ending invariants (which encode the asymptotic geometry of their ends.) The Tameness Theorem thus implies that we have a complete classification of all hyperbolic 3-manifolds with finitely generated fundamental group. A complete discussion of this classification theorem is outside the scope of this paper but we will try to give a rough idea of the result. (See Minsky's survey paper [74] for a more intensive exposition of the classificatiion theorem.)

Let N be a hyperbolic 3-manifold with finitely generated fundamental group and let C be a compact core for N. We may assume that $N - C$ is homeomorphic to $\partial C \times (0, \infty)$. For simplicity, we will assume in our discussion that N has no cusps.

If a component S of ∂C abuts a geometrically finite end of N, then it inherits a conformal structure from the corresponding (homeomorphic) component $\bar{S}$ of the conformal boundary and this conformal structure is the ending invariant associated to that end. If U is the component of $N - C(N)$ bounded by $\bar{S}$, then there is a natural map $r : U \to \partial C(N)$ given by taking a point in U to the nearest point on $\partial C(N)$. If $x \in U$, then every point on the geodesic ray beginning at $r(x)$ and passing through x is taken to $r(x)$, so this map induces a product structure on U. One can check, see Epstein-Marden [43], that U is homeomorphic to $\bar{S} \times (0, \infty)$ and that the metric on U is bilipschitz to $\cosh^2(t) ds^2 + dt^2$ where t is the real coordinate and ds^2 is the Poincaré metric on $\bar{S}$. So, one sees very precisely that the conformal structure on $\bar{S}$ encodes the geometry of the associated geometrically infinite end.

If a component S of ∂C abuts a simply degenerate end of N, then there exists a sequence of simplicial hyperbolic surface $\{f_n : S \to N\}$ exiting the end. There exists a uniform constant B, such that any metric induced on S by a simplicial hyperbolic surface contains a simple closed geodesic of length at most B. For all n, let α_n be a simple closed curve on S which has length at most B in the metric on S induced by f_n. In this situation, the *ending lamination* of the end associated to S is the "limit" λ of the sequence $\{\alpha_n\}$.

We may make sense of the limit in two equivalent ways. In the first method, we fix a hyperbolic structure on S and let λ be the Hausdorff limit of the sequence $\{\alpha_n^*\}$ of geodesic representatives of the α_n. This limit is a closed set which is a disjoint union of simple geodesics, i.e. a geodesic lamination. (To be more precise, we must remove any isolated leaves in the resulting lamination, to ensure that it is well-defined.) The second method involves considering the curve complex $\mathcal{C}(S)$. The vertices of the curve complex are isotopy classes of simple closed curves

and we say that a collection of vertices spans a simplex if and only if they have a collection of disjoint representatives. The curve complex is locally infinite, but Masur and Minsky [68] proved that it is Gromov hyperbolic. We may then define λ to be the point in the Gromov boundary which is the limit of the vertices associated to the sequence $\{\alpha_n\}$.

Bonahon [12] and Thurston [101] proved that the ending lamination is well-defined for simply degenerate ends of hyperbolic 3-manifold with finitely generated, freely indecomposable fundamental group and Canary [29] showed that they can be defined for simply degenerate ends of topologically tame hyperbolic 3-manifolds. Klarreich [60], see also Hamenstadt [49], proved that one can identify the Gromov boundary of the curve complex with the set of potential ending laminations.

The ending invariants of N are encoded by the compact core C where each boundary component of C is equipped with either a conformal structure or an ending lamination. Thurston conjectured that this information determined N up to isometry. Brock, Canary and Minsky [75, 21, 22] proved this conjecture for topologically tame hyperbolic 3-manifolds in a proof which builds on earlier work of Masur and Minsky [68, 69]. The resolution of Marden's Tameness Conjecture gives the following:

Ending Lamination Theorem: *A hyperbolic 3-manifold with finitely generated fundamental group is determined up to isometry by its ending invariants.*

Alternate approaches to this result are given by Bowditch [15], Brock-Bromberg-Evans-Souto [20], and Rees [87]. Minsky [73] earlier established the Ending Lamination Theorem for punctured torus groups.

We remark that one can determine exactly which end invariants arise, so this is a complete classification theorem(see Ohshika [80] for the characterization in the case that the fundamental group is freely indecomposable.)

One consequence of the proof of the Ending Lamination Theorem is the following common generalization of Mostow [78] and Sullivan's [97] rigidity theorems.

Corollary 10.1. *Let G be a finitely generated, torsion-free group which is not virtually abelian. If two discrete faithful representations $\rho_1 : G \to \mathbb{PSL}_2(\mathbb{C})$ and $\rho_2 : G \to \mathbb{PSL}_2(\mathbb{C})$ are conjugate by an orientation-preserving homeomorphism φ of $\widehat{\mathbb{C}}$, then they are quasiconformally conjugate. Moreover, if φ is conformal on $\Omega(\rho_1(G))$, then φ is conformal.*

11. Deformation Theory of hyperbolic 3-manifolds

It is natural to consider the space of all (marked) hyperbolic 3-manifolds of fixed homotopy type. One may think of this as a 3-dimensional generalization of Teichmüller theory. The Mostow-Prasad Rigidity Theorem [78, 86] assures us that if a hyperbolic 3-manifold N has finite volume, then any homotopy equivalence of N to another hyperbolic 3-manifold is homotopic to an isometry, so we will only consider the deformation theory of infinite volume hyperbolic manifolds.

Let M be a compact, atoroidal, irreducible 3-manifold with a non-toroidal boundary component. We consider the space $AH(M)$ of (marked) hyperbolic 3-manifolds homotopy equivalent to M. Formally, we define

$$AH(M) = \{\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C}) | \ \rho \text{ discrete and faithful}\}/\mathrm{PSL}_2(\mathbf{C}).$$

The deformation space sits as a subset of the character variety

$$X(M) = Hom_T(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))//\mathrm{PSL}_2(\mathbf{C})$$

where $Hom_T(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))$ denotes the space of discrete faithful representations $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$ with the property that if g is a non-trivial element of a rank two abelian subgroup of $\pi_1(M)$, then $\rho(g)$ is parabolic. An element $\rho \in AH(M)$ gives rise to a pair (N_ρ, h_ρ) where $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$ is a hyperbolic 3-manifold and $h_\rho : M \to N_\rho$ is a homotopy equivalence. (We could alternatively have defined $AH(M)$ to be the set of such pairs up to appropriate equivalence.)

Marden [65] and Sullivan [98] proved that the interior of $AH(M)$ (as a subset of $X(M)$) consists of geometrically finite representations such that $\rho(g)$ is parabolic if and only if g is a non-trivial element in a free abelian subgroup of $\pi_1(M)$ of rank two. The classical deformation theory of Kleinian groups tells us that geometrically finite points in $AH(M)$ are determined by their homeomorphism type and the conformal structures on their boundary (see Bers [10] or Canary-McCullough [35] for a survey of this theory.)

To state the parameterization theorem for $\mathrm{int}(AH(M))$ we need a few definitions. We first define $\mathcal{A}(M)$ to be the set of oriented, compact, irreducible, atoroidal (marked) 3-manifolds homotopy equivalent to M. More formally, $\mathcal{A}(M)$ is the set of pairs (M', h') where M' is an oriented, compact, irreducible, atoroidal 3-manifold and $h' : M \to M'$ is a homotopy equivalence where two pairs (M_1, h_1) and (M_2, h_2) are considered equivalent if there exists an orientation-preserving homeomorphism $j : M_1 \to M_2$ such that $j \circ h_1$ is homotopic to h_2. We define $Mod_0(M')$ to be the group of isotopy classes of homeomorphisms of M' which are homotopic to the identity. We define $\partial_{NT} M'$ to be the

non-toroidal components of ∂M and we let $\mathcal{T}(\partial_{NT}M')$ denote the Teichmüller space of all (marked) conformal structures on $\partial_{NT}M'$.

Theorem 11.1. *(Ahlfors, Bers, Kra, Marden, Maskit, Sullivan,Thurston)*

$$\text{int}(AH(M)) \cong \bigcup_{(M,h')'\in\mathcal{A}(M)} \mathcal{T}(\partial_{NT}M')/Mod_0(M')$$

In particular, we see that the components of $\text{int}(AH(M))$ are in one-to-one correspondence with elements of $\mathcal{A}(M)$. Moreover, Maskit [67] showed that $Mod_0(M')$ always acts freely on $\mathcal{T}(\partial_{NT}M')$, so each component is a manifold. Although, see McCullough [70], $Mod_0(M')$ is often infinitely generated, so the fundamental group of a component can be infinitely generated. However, if $\pi_1(M)$ is freely indecomposable, then $Mod_0(M')$ is trivial for all $(M',h')\in\mathcal{A}(M)$, so in this case each component is topologically an open ball.

Bers, Sullivan and Thurston conjectured that $AH(M)$ is the closure of its interior. More concretely, this predicts that every hyperbolic 3-manifold with finitely generated fundamental group is an (algebraic) limit of a sequence of geometrically finite hyperbolic 3-manifolds. The Tameness Theorem is a crucial tool in the recent proof of this conjecture.

Density Theorem: *If M is a compact hyperbolizable 3-manifold, then $AH(M)$ is the closure of its interior* $\text{int}(AH(M))$.

One may derive the Density Theorem from the Tameness Theorem, the Ending Lamination Theorem, and convergence results of Thurston [102, 103], Kleineidam-Souto [61], Lecuire [64] and Kim-Lecuire-Ohshika [59]. Basically, the idea here is to consider the end invariants of a given 3-manifold, use the convergence results to construct a hyperbolic 3-manifold with the given end invariants which arises as a limit of geometrically finite hyperbolic 3-manifolds. (In the case that the manifold is homotopy equivalent to a compression body one must use clever arguments of Namazi-Souto [79] or Ohshika [82] to verify that our limits have the correct ending invariants.) One then applies the Ending Lamination Theorem to show that our limit and our original manifold are the same.

The other approach makes use of the deformation theory of cone-manifolds developed by Hodgson-Kerckhoff [51, 52] and Bromberg [24]. Many cases of the Density Theorem were established by Bromberg and Brock-Bromberg [18] and their approach was generalized to prove the entire theorem by Bromberg and Souto [26]. This approach uses the Tameness Theorem but not the Ending Lamination Theorem.

Anderson, Canary and McCullough [8] gave a complete enumeration of the components of the closure of $\text{int}(AH(M))$ in the case that $\pi_1(M)$ is freely indecomposable. Given the resolution of the Density Conjecture, we now have a complete enumeration of the components of $AH(M)$.

Again, to state the result, we will need more definitions. Given two compact irreducible 3-manifolds M_1 and M_2 with nonempty incompressible boundary, a homotopy equivalence $h\colon M_1 \to M_2$ is a *primitive shuffle equivalence* if there exists a finite collection V_1 of primitive solid torus components of $\Sigma(M_1)$ and a finite collection V_2 of solid torus components of $\Sigma(M_2)$, so that $h^{-1}(V_2) = V_1$ and so that h restricts to an orientation-preserving homeomorphism from the closure of $M_1 - V_1$ to the closure of $M_2 - V_2$. We recall that a solid torus V in the characteristic submanifold is said to be *primitive* if given any annulus component A of $V \cap \partial M$ the image of $\pi_1(A)$ in $\pi_1(M)$ is a maximal abelian subgroup. (We refer the reader to Jaco-Shalen [54] or Johannson [55] for a discussion of the characteristic submanifold and to Canary-McCullough [35] for a discussion of the characteristic submanifold in the setting of hyperbolizable 3-manifolds.) Primitive shuffle equivalence induces a finite-to-one equivalence relation on $\mathcal{A}(M)$ and we let $\widehat{\mathcal{A}}(M)$ denote the quotient.

Anderson, Canary and McCullough [8] prove that if $\pi_1(M)$ is freely indecomposable, then two components of $\text{int}(AH(M))$ have intersecting closure if and only if their associated (marked) homeomorphism types differ by a primitive shuffle equivalence. Combining this result with the Density Theorem we obtain:

Theorem 11.2. *If M is a compact, hyperbolizable 3-manifold with freely indecomposable fundamental group, then the components of $AH(M)$ are in one-to-one correspondence with $\widehat{\mathcal{A}}(M)$.*

Canary and McCullough [35] proved that if $\pi_1(M)$ has freely indecomposable fundamental group, then $\mathcal{A}(M)$ has infinitely many elements if and only if M has double trouble. M is said to have *double trouble* if there exist simple closed curves α, β and γ in ∂M which are homotopic in M, but not in ∂M, and α and β lie on non-toroidal boundary components of M, while γ lies on a toroidal boundary component. (Canary and McCullough [35] give a complete analysis of when $\mathcal{A}(M)$ is finite in the general case.) So, we obtain the following corollary:

Corollary 11.3. *Let M be a compact, hyperbolizable 3-manifold with freely indecomposable fundamental group. Then, $AH(M)$ has infinitely many components if and only if M has double trouble.*

Remark: The author has recently completed a survey article [32] on the deformation theory of hyperbolic 3-manifolds which contains a more detailed discussion of the topic.

References

[1] I. Agol, "Tameness of hyperbolic 3-manifolds," preprint.

[2] I. Agol, "Criteria for virtual fibering," preprint.

[3] I. Agol, D. Long and A. Reid, "The Bianchi groups are separable on geometrically finite subgroups," *Ann. of Math.* **153** (2001), 599–621.

[4] L. Ahlfors, "Fundamental polyhedrons and limit point sets of Kleinian groups," *Proc. Nat. Acad. Sci. USA* **55** (1966), 251–254.

[5] L. Ahlfors, "Finitely generated Kleinian groups," *Amer. J. Math.* **86** (1964), 413–429.

[6] J. W. Anderson and R. D. Canary, "Cores of hyperbolic 3-manifolds and limits of Kleinian groups," *Amer. J. Math.* **118** (1996), 745–779.

[7] J. W. Anderson, R. D. Canary, M. Culler, and P. Shalen, "Free Kleinian groups and volumes of hyperbolic 3-manifolds," *J. Diff. Geom.* **44** (1996), 738–782.

[8] J. W. Anderson, R. D. Canary and D. McCullough, "On the topology of deformation spaces of Kleinian groups," *Annals of Math.* **152** (2000), 693-741.

[9] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry*, Springer-Verlag Universitext, 1992.

[10] L. Bers, "On moduli of Kleinian groups," *Russian Math. Surveys* **29** (1974), 88–102.

[11] C. Bishop and P. Jones, "Hausdorff dimension and Kleinian groups," *Acta Math.* **179** (1997), 1–39.

[12] F. Bonahon, "Bouts des variétés hyperboliques de dimension 3," *Annals of Math.* **124** (1986), 71–158.

[13] B. Bowditch, "Geometrical finiteness for hyperbolic groups," *J. Funct. Anal.* **113** (1993), 245–317.

[14] B. Bowditch, "Notes on tameness," preprint.

[15] B. Bowditch, "Geometric models for hyperbolic 3-manifolds," preprint.

[16] P. Braam, "A Kaluza-Klein approach to hyperbolic three-manifolds," *L'Enseign. Math.* **34**(1988), 275–311.

[17] M. Bridgeman and R. Canary, "From the boundary of the convex core to the conformal boundary," *Geom. Ded.* **96**(2003), 211–240.

[18] J. Brock and K. Bromberg, "On the density of geometrically finite Kleinian groups," *Acta Math.* **192**(2004), 33–93.

[19] J. Brock, K. Bromberg, R. Evans and J. Souto, "Tameness on the boundary and Ahlfors' measure conjecture," *Publ. Math. I.H.E.S.* **98** (2003), 145–166.

[20] J. Brock, K. Bromberg, R. Evans and J. Souto, in preparation.

[21] J. Brock, R.Canary and Y. Minsky, "The classification of Kleinian surface groups II: the ending lamination conjecture," preprint.

[22] J. Brock, R.Canary and Y. Minsky, "The classification of finitely generated Kleinian groups," in preparation.

[23] J. Brock and J. Souto, "Algebraic limits of geometrically finite manifolds are tame," *Geom. Funct. Anal.* **16** (2006), 1–39.

[24] K.Bromberg, "Rigidity of geometrically finite hyperbolic cone-manifolds," *Geom. Ded.* **105**(2004), 143–170.

[25] K. Bromberg, "Projective structures with degenerate holonomy and the Bers density conjecture," *Annals of Math.*, to appear.

[26] K. Bromberg and J. Souto, "The density conjecture: a prehistoric approach," in preparation

[27] D. Calegari and D. Gabai, "Shrinkwrapping and the taming of hyperbolic 3-manifolds," *J.A.M.S.* **19**(2006), 385–446.

[28] R.D. Canary, "On the Laplacian and the geometry of hyperbolic 3-manifolds," *J. Diff. Geom.*, **36**(1992), 349–367.

[29] R. D. Canary, "Ends of hyperbolic 3-manifolds," *J. Amer. Math. Soc.* **6** (1993), 1–35.

[30] R.D. Canary, "Covering theorems for hyperbolic 3-manifolds," *Proceedings of Low-Dimensional Topology*, International Press, 1994, 21-30.

[31] R.D. Canary, "A covering theorem for hyperbolic 3-manifolds and its applications," *Topology* **35**(1996), 751–778.

[32] R. D. Canary, "Introductory bumponomics: deformation theory of hyperbolic 3-manifolds," preprint.

[33] R.D. Canary, D.B.A. Epstein, and P. Green, "Notes on notes of Thurston", in *Analytical and Geometrical Aspects of Hyperbolic Spaces*, Cambridge University Press, 1987, 3–92.

[34] R.D. Canary and C. Leininger, "Kleinian groups with discrete length spectrum," *Bull. L.M.S.*, **39**(2007), 189–193.

[35] R.D. Canary and D. McCullough, "Homotopy equivalences of 3-manifolds and deformation theory of Kleinian groups," *Mem. A.M.S.*, vol. **172**(2004), no. 812.

[36] R. D. Canary and Y. N. Minsky, "On limits of tame hyperbolic 3-manifolds," *J. Diff. Geom.* **43** (1996), 1–41.

[37] R.D. Canary and E.C. Taylor, "Kleinian groups with small limit sets," *Duke Math. J.* **73** (1994), 371–381.

[38] H. Cao and X. Zhu, "A complete proof of the Poincaré and Geometrization conjectures–Application of the Hamilton-Perelman theory of the Ricci flow," *Asian J. Math.* **10**(2006), 169–492.

[39] S. Choi, "The PL-methods for hyperbolic 3-manifolds to prove tameness," preprint.

[40] V. Chuckrow, "On Schottky groups with applications to Kleinian groups," *Ann. of Math.* **88**(1968). 47–61.

[41] M. Culler and P.B. Shalen, "Paradoxical decompositions, 2-generator Kleinian groups, and volumes of hyperbolic 3-manifolds," *Jour. A.M.S.* **5**(1992), 231–288.

[42] M. Culler and P.B. Shalen, "Singular surfaces, mod 2 homology and hyperbolic volume," preprint.

[43] D.B.A. Epstein and A. Marden, "Convex hulls in hyperbolic spaces, a theorem of Sullivan, and measured pleated surfaces" in *Analytical and Geometrical Aspects of Hyperbolic Spaces*, Cambridge University Press, 1987, 113–253.

[44] R. Evans, "Tameness persists in type-preserving strong limits," preprint.

[45] R. Gitik, "Doubles of groups and hyperbolic LERF 3-manifolds," *Ann. of Math.* **150** (1999), 775–806.

[46] T. Gelander and Y. Glasner, "Countable primitive groups," *Geom. Funct. Anal.*, to appear.

[47] Y. Glasner, J. Souto and P. Storm, "Finitely generated subgroups of lattices in $\mathbb{PSL}_2(\mathbb{C})$," preprint.

[48] L. Greenberg, "Fundamental polyhedra for Kleinian groups," *Ann. of Math.* **84** (1966), 433–441.

[49] U. Hamenstadt, "Train tracks and the Gromov boundary of the complex of curves," preprint.

[50] J. Hempel, "The finitely generated intersection property for Kleinian groups," in *Combinatorial Group Theory and Topology*, ed. by S. Gersten and J. Stallings, Princeton University Press, 1987, 18–24.

[51] C.D. Hodgson and S.P. Kerckhoff, Steven, "Rigidity of hyperbolic cone-manifolds and hyperbolic Dehn surgery," *J. Diff. Geom.* **48**(1998), 1–59.

[52] C. Hodgson and S.P. Kerckhoff, "Universal bounds for hyperbolic Dehn surgery," *Ann. of Math.* **162**(2005), 367–421.

[53] W. Jaco, *Lectures on 3-manifold Topology*, CBMS no. 43, American Mathematical Society, 1980.

[54] W. Jaco and P. Shalen "Seifert fibered spaces in 3-manifolds," *Mem. Amer. Math. Soc.* **220** (1979).

[55] K. Johannson, *Homotopy Equivalences of 3-manifolds with Boundary*, Lecture Notes in Mathematics, vol. 761, Springer-Verlag, 1979.

[56] T. Jørgensen, "On discrete groups of Möbius transformations," *Amer. J. of Math.* **98**(1976), 739–49.

[57] T. Jørgensen, "Compact 3-manifolds of constant negative curvature fibering over the circle," *Ann. Math.* **106** (1977), 61–72.

[58] M. Kapovich, *Hyperbolic Manifolds and Discrete groups: Lectures on Thurston's Hyperbolization*, Birkhauser, 2000.

[59] I. Kim, C. Lecuire and K. Ohshika, "Convergence of freely decomposable Kleinian groups," preprint.

[60] E. Klarreich, "The boundary at infinity of the curve complex and the relative Teichmüller space, " preprint.

[61] G. Kleineidam and J. Souto, "Algebraic convergence of function groups," *Comment. Math. Helv.* **77** (2002), 244-269.

[62] G. Kleineidam and J. Souto, "Ending laminations in the Masur domain," in *Kleinian groups and hyperbolic 3-manifolds (Warwick, 2001)*, ed. by Y. Komori, V. Markovic and C. Series, *London Math. Soc. Lecture Notes* **299**(2003), 105–129.

[63] B. Kleiner and J. Lott, "Notes on Perelman's papers," preprint.

[64] C. Lecuire, "An extension of Masur domain," in *Spaces of Kleinian groups*, ed. by Y. Minsky, M. Sakuma and C. Series, *London Math. Soc. Lecture Notes* **329**(2006), 49–73.

[65] A. Marden, "The geometry of finitely generated Kleinian groups," *Annals of Math.* **99** (1974), 383–462.

[66] G. Margulis and G. Soifer, "Maximal subgroups of infinite index in finitely generated linear groups," *J. Algebra* **69**(1981), 1–23.

[67] B. Maskit, "Self-maps of Kleinian groups," *Amer. J. Math.* **93** (1971), 840–856.

[68] H. A. Masur and Y. Minsky, "Geometry of the complex of curves I: Hyperbolicity, " *Invent. Math.* **138** (1999), 103–149.

[69] H.A. Masur and Y.N. Minsky, "Geometry of the complex of curves II: Hierarchical structure," *Geom. Funct. Anal.* **10** (2000), 902–974.

[70] D. McCullough, "Twist groups of compact 3-manifolds," *Topology* **24** (1985), 461–474.

[71] D. McCullough, A. Miller and G. A. Swarup, "Uniqueness of cores of noncompact 3-manifolds," *J. London Math. Soc* **32** (1985), 548–556.

[72] Y. Minsky, "Harmonic maps into hyperbolic 3-manifolds," *Trans. A.M.S.* , **332** (1992), 607–632.

[73] Y.N. Minsky, "The classification of punctured torus groups," *Annals of Math.*, **149** (1999), 559–626.

[74] Y.N. Minsky, "End invariants and the classification of hyperbolic 3-manifolds," in *Current developments in mathematics, 2002*, International Press (2003), 181–217.

[75] Y.N. Minsky, "The classification of Kleinian surface groups I: models and bounds," *Annals of Math.*, to appear.

[76] J. W. Morgan, "On Thurston's uniformization theorem for three-dimensional manifolds," in *The Smith Conjecture*, ed. by J. Morgan and H. Bass, Academic Press, 1984, 37–125.

[77] J. Morgan and G. Tian, "Ricci flow and the Poincaré conjecture," preprint.

[78] G. D. Mostow, "Quasiconformal mappings in n-space and the rigidity of hyperbolic space forms," *Publ. I.H.E.S.* **34** (1968), 53–104.

[79] H. Namazi and J. Souto, "Non-realizability and ending laminations," in preparation.

[80] K. Ohshika, "Ending laminations and boundaries for deformation spaces of Kleinian groups," *J. London Math. Soc.* **42** (1990), 111–121.

[81] K. Ohshika, "Kleinian groups which are limits of geometrically finite groups," *Mem. A.M.S.* **177** (2005), no. 834.

[82] K. Ohshika, "Realising end invariants by limits of convex cocompact groups," preprint.

[83] S.J. Patterson, "Lectures on limit sets of Kleinian groups," in *Analytical and Geometrical Aspects of Hyperbolic Spaces*, Cambridge University Press, 1987, 281–323.

[84] G. Perelman, "The entropy formula for the Ricci flow and its geometric applications," preprint

[85] G. Perelman, "Ricci flow with surgery on three-manifolds," preprint.

[86] G. Prasad, "Strong rigidity of **Q**-rank 1 lattices," *Invent. Math.* **21** (1973), 255–286.

[87] M. Rees, "The ending laminations theorem direct from Teichmüller geodesics," preprint.

[88] H. Rubinstein and S. Wang, "π_1-injective surfaces in graph manifolds." *Comm. Math. Helv.* **73** (1998), 499–515.

[89] P. Scott, "Compact submanifolds of 3-manifolds," *J. London Math. Soc.* **7** (1974), 246–250.

[90] G. P. Scott, "Subgroups of surface groups are almost geometric," *J. London Math. Soc.* **17** (1978), 555–565.

[91] G.P. Scott, Correction to"Subgroups of surface groups are almost geometric," *J. London Math. Soc.* **32** (1985), 217–220.

[92] P. Scott and T. Tucker, "Some examples of exotic non-compact 3-manifolds," *Quart. J. Math. Oxford* **40**(1989), 481–499.

[93] J. Simon, "Compactification of covering spaces of compact 3-manifolds.," *Mich. Math. J.* **23** (1976), 245–256.

[94] T. Soma, "Existence of ruled wrappings in hyperbolic 3-manifolds," *Geom. Top.* **10** (2006), 1173–1184

[95] T. Soma, "3-manifold groups with the finitely generated intersection property," *Trans. A.M.S.* **331**(1992), 761–769.

[96] J. Souto, "A note on the tameness of hyperbolic 3-manifolds," *Topology* **44**(2005), 459–474.

[97] D. Sullivan, "The density at infinity of a discrete group of hyperbolic motions," *Publ. I.H.E.S.* **50**(1979), 419–450.

[98] D. Sullivan, "Quasiconformal homeomorphisms and dynamics. II. Structural stability implies hyperbolicity for Kleinian groups," *Acta Mathematica* **155** (1985), 243–260.

[99] D. Sullivan, "Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups," *Acta. Math.* **153**(1984), 259–277.

[100] P. Susskind, "Kleinian groups with intersecting limit sets," *Journal d'Analyse Mathematique* **52**(1989), 26–38.

[101] W. P. Thurston, *The Geometry and Topology of Three-Manifolds*, Princeton University course notes, available at http://www.msri.org/publications/books/gt3m/ (1980).

[102] W.P. Thurston, "Hyperbolic Structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle," preprint.

[103] W.P. Thurston, "Hyperbolic Structures on 3-manifolds, III: Deformations of 3-manifolds with incompressible boundary," preprint.

[104] T.W. Tucker, "Non-compact 3-manifolds and the missing-boundary problem," *Topology* **13**(1974), 267–73.

[105] P. Tukia, "On the dimension of limit sets of geometrically finite Möbius groups," *Ann. Acad. Sci. Fenn.* **19** (1994), 11–24.

[106] J.H.C. Whitehead, "Some examples of exotic noncompact 3-manifolds," *Quart. J. Math. Oxford Ser.* **40** (1989), 481–499.

[107] D. Wise, "Subgroup separability of the figure 8 knot group," *Topology* **45** (2006), 421–463.

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